hep-th@xxx/9512079

# Singular Vectors of the Topological Conformal Algebra

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A general construction is found for 'topological' singular vectors of the twisted $N=2$ superconformal algebra. It demonstrates many parallels with the known construction for affine $s\ell(2)$ singular vectors due to Malikov–Feigin–Fuchs, but is formulated independently of the latter. The two constructions taken together provide an isomorphism between the topological and affine $s\ell(2)$ singular vectors. The general formula for topological singular vectors can be reformulated as a chain of direct recursion relations that allow one to derive a given singular vector $|S(r,s)\rangle$ from the lower ones $|S(r,s'<s)\rangle$. We also introduce generalized Verma modules over the twisted $N=2$ algebra and show that they provide a natural setup for the new construction for topological singular vectors.



## 1 Introduction

Singular vectors in Verma modules of infinite-dimensional algebras represented in conformal models play an important role both in conformal field theory and in the representation theory. Their significance in conformal models was appreciated already in [1], where decoupling conditions for singular vectors were shown to lead to differential equations on correlators. Mathematically, this is a manifestation of the fact that irreducible representations are obtained by factoring the Verma modules over submodules built on singular vectors [2]. 'Positions' of singular vectors determine embeddings of Verma modules. A number of papers have been devoted to finding positions as well as explicit representations, and various ways to derive, singular vectors in different models [3]–[20].

A closed formula for all singular vectors in Verma modules over $s\ell(n)$ (in particular, $s\ell(2)$) Kač–Moody algebras was found in [7], and it has stimulated efforts aimed at deriving singular vectors for other popular algebras (Virasoro, $N=2$, $W_3, \ldots$) related to affine Lie algebras. Morphisms between [conformal field theories with] different algebras have been used to find mappings between the corresponding singular states. For instance, hamiltonian reduction of $s\ell(2)$ and $s\ell(3)$ affine algebras allows one to reduce the corresponding Kač–Moody singular vectors to those in the appropriate 'matter' theories. These reductions, however, do not provide *intrinsic* constructions for singular vectors in algebras other than the affine Lie algebras. A challenging problem is to find generating constructions for singular vectors using the idea of 'analytic' (in fact, *algebraic*) continuation developed in terms of a given algebra only.

The aim of the present paper is to give a complete realization of such a program for the so called *topological* singular vectors, i.e. BRST- ($\mathcal{Q}_0$-) or $\mathcal{G}_0$- invariant singular vectors of the twisted $N=2$ superconformal algebra, built upon BRST-invariant *chiral* primary states. We will present a closed algebraic construction for all topological singular vectors in Verma modules. This construction is in fact a particular case of a more general one that produces singular vectors in *generalized* Verma modules.

It will be seen that the proposed construction has many similarities with the MFF one, although it relies on more involved algebraic properties. It is indeed motivated by the results of [18], which strongly suggest an isomorphism between $s\ell(2)$- and topological singular vectors. One should recall the apparent simplicity of the MFF formula, in which an involved algebraic expression is 'packed' into a single *monomial*; that monomial, however, does not look like an element of a Verma module, and a set of algebraic transformations is required in order to bring it to the 'Verma' form. It is in the course of this rearrangement that complexity of the expression grows rapidly.

Just like the MFF formula, the general formula for topological singular vectors (eqs. (5.3) and (5.7)) looks deceptively simple, but it also does require certain algebraic manipulations in order to be

transformed into a 'Verma' form. It involves objects that look like a continuation of products of modes of the fermionic operators $\mathcal{G}$ and $\mathcal{Q}$ ($G$ and $\overline{G}$ or $G^{\pm}$ in a different nomenclature) to a non-integral number of factors. These objects will be called intertwining[1] operators; they map between modules related by the spectral flow transform [24, 25]. The appearance of the intertwining operators is very natural since any singular vector in a given module can be viewed as an intertwiner of that module with itself. We arrive at the general construction for topological singular vectors by simply looking for the appropriate intertwiners[2]. Then, to compute a singular vector in the Verma form, we will need a set of commutation relations between the intertwiners and the generators of the topological conformal algebra.

A very useful reformulation of the construction for topological singular vectors can be given in the form of a sequence of recursion relations between topological singular vectors of different levels. That is, given a singular vector $|S(r,s)\rangle^{\pm}$, there is a unique procedure of building the singular vector $|S(r,s+1)\rangle^{\mp}$. Our construction can thus be reformulated as a method to derive a given topological singular vector from a 'lower' one. Such a recursive reformulation is very convenient for computational purposes, but it can also be used to prove by induction that the construction does indeed give singular vectors. In whatever reformulation, the proposed construction is *direct* in that it does not require solving 'inverse' problems such as e.g. finding a solution of (however simple) system of equations.

We begin in the next section with setting up the problem and then recall in section 3 the MFF formula for the affine $s\ell(2)$ singular vectors and the Kazama–Suzuki mapping. Analogies with the $s\ell(2)$ case are quite useful in motivating the subsequent construction, but the properties of the intertwiners necessary in the topological case have to be formulated independently, which we do in section 4. Then, in section 5, we give the general formula for topological singular vectors. The algebraic scheme that allows one to rewrite these as elements of the usual Verma modules is described in section 6. The recursive reformulation is given in section 7, where we explain how topological singular vectors on different levels are related by purely algebraic manipulations. Section 8 contains several concluding remarks.

The main results of this paper have been announced in [31].

## 2 Topological conformal algebra and singular vectors

By the topological [conformal] algebra (see [26, 27]) we will mean the twisted $N=2$ superconformal algebra spanned by the integer-moded generators $\mathcal{L}_m$ (Virasoro generators), $\mathcal{Q}_m$ (BRST current), $\mathcal{H}_m$ ($U(1)$ current), and $\mathcal{G}_m$ (fermionic spin-2 current):

$$\begin{aligned}
[\mathcal{L}_m,\mathcal{L}_n] &= (m-n)\mathcal{L}_{m+n}\,, & [\mathcal{H}_m,\mathcal{H}_n] &= \frac{\mathsf{c}}{3}m\delta_{m+n,0}\,,\\
[\mathcal{L}_m,\mathcal{G}_n] &= (m-n)\mathcal{G}_{m+n}\,, & [\mathcal{H}_m,\mathcal{G}_n] &= \mathcal{G}_{m+n}\,,\\
[\mathcal{L}_m,\mathcal{Q}_n] &= -n\mathcal{Q}_{m+n}\,, & [\mathcal{H}_m,\mathcal{Q}_n] &= -\mathcal{Q}_{m+n}\,, \qquad m,\ n\in\mathbb{Z}\,.\\
[\mathcal{L}_m,\mathcal{H}_n] &= -n\mathcal{H}_{m+n}+\frac{\mathsf{c}}{6}(m^2+m)\delta_{m+n,0}\,,\\
\{\mathcal{G}_m,\mathcal{Q}_n\} &= 2\mathcal{L}_{m+n}-2n\mathcal{H}_{m+n}+\frac{\mathsf{c}}{3}(m^2+m)\delta_{m+n,0}\,,
\end{aligned} \tag{2.1}$$

The singular vectors we are interested in are built on 'topological primary states', which are defined

[1] Strictly speaking, these operators do not commute with the action of the algebra and thus are not intertwiners, but they can be realized as such by integral operators on the appropriate supermanifold in the spirit of [23]. We will therefore use the convenient name 'intertwiners' for these operators in the future.

[2] In the MFF case [7], the 'intertwiners' can be written as $(J_0^+)^p$ and $(J_{-1}^-)^p$, $p\in\mathbb{C}$, which makes some of their algebraic properties self-evident. In the $N=2$ case, more relations will have to be formulated explicitly, which is one of the reasons for the construction to appear more involved.

by a set of annihilation conditions (to be called the 'highest-weight' conditions for brevity) that can be described as follows. First, one imposes the conditions

$$\mathcal{Q}_{\geq 0}|\mathsf{h}\rangle_{\rm top} = \mathcal{G}_{\geq 1}|\mathsf{h}\rangle_{\rm top} = \mathcal{L}_{\geq 1}|\mathsf{h}\rangle_{\rm top} = \mathcal{H}_{\geq 1}|\mathsf{h}\rangle_{\rm top} = 0 \tag{2.2}$$

(the $\mathcal{Q}_0$-condition is sometimes referred to as BRST-invariance). In addition, one imposes chirality [25], which in the present, twisted, version amounts to

$$\mathcal{G}_0|\mathsf{h}\rangle_{\rm top} = 0\,. \tag{2.3}$$

This implies then that the eigenvalue of $\mathcal{L}_0$ vanishes, hence the 'Cartan' part of the highest-weight conditions reads

$$\mathcal{L}_0|\mathsf{h}\rangle_{\rm top} = 0\,, \qquad \mathcal{H}_0|\mathsf{h}\rangle_{\rm top} = \mathsf{h}|\mathsf{h}\rangle_{\rm top} \tag{2.4}$$

Equations (2.2)–(2.4) determine what we will call the topological primary states; $\mathsf{h}$ will be called the topological $U(1)$ charge.

A BRST-invariant topological singular vector built on a state $|\mathsf{h}\rangle_{\rm top}$ is required to satisfy the same annihilation conditions as in (2.2),

$$\mathcal{Q}_{\geq 0}|S\rangle^+ = \mathcal{G}_{\geq 1}|S\rangle^+ = \mathcal{L}_{\geq 1}|S\rangle^+ = \mathcal{H}_{\geq 1}|S\rangle^+ = 0\,. \tag{2.5}$$

This determines a half of singular vectors that can be built on the topological primary states, the other half being determined by

$$\mathcal{Q}_{\geq 1}|S\rangle^- = \mathcal{G}_{\geq 0}|S\rangle^- = \mathcal{L}_{\geq 1}|S\rangle^- = \mathcal{H}_{\geq 1}|S\rangle^- = 0\,. \tag{2.6}$$

These singular vectors exist in a Verma module $V_{\mathsf{h}}$ over $|\mathsf{h}\rangle_{\rm top}$ if $\mathsf{h} = \mathsf{h}^+(r,s)$ or $\mathsf{h} = \mathsf{h}^-(r,s)$ respectively [5, 18], where

$$\begin{array}{rcl}\mathsf{h}^+(r,s) & = & -\frac{r-1}{t} + s - 1\,,\\ \mathsf{h}^-(r,s) & = & \frac{r+1}{t} - s\,,\end{array} \qquad r,s\in\mathbb{N} \tag{2.7}$$

and $t \equiv k+2$ is related to the topological central charge $\mathsf{c}$ by $\mathsf{c} = 3(t-2)/t$. Our aim is to find an explicit formula for these singular vectors for arbitrary $r$ and $s\in\mathbb{N}$.

Denote by $V^+_{r,s}$ and $V^-_{r,s}$ respectively the Verma modules built on the topological primary states with topological $U(1)$ charges (2.7). We will mainly concentrate for definiteness on the singular vectors, denoted as $|S(r,s)\rangle^+$, in the '+'-modules $V^+_{r,s}$. Let us note right now that all such vectors have the structure

$$|S(r,s)\rangle^+ = \mathcal{Q}_0\ldots\mathcal{Q}_{r-1}\cdot|T(r,s)\rangle^+\,, \tag{2.8}$$

where $|T(r,s)\rangle^+$ is a Verma module element that, too, satisfies certain 'highest-weight' conditions

$$\begin{array}{l}\mathcal{Q}_{\geq r}|T(r,s)\rangle^+ = \mathcal{G}_{\geq -r}|T(r,s)\rangle^+ = \mathcal{L}_{\geq 1}|T(r,s)\rangle^+ = \mathcal{H}_{\geq 1}|T(r,s)\rangle^+ = 0\,,\\ \mathcal{H}_0|T(r,s)\rangle^+ = (\mathsf{h}^+(r,s)+r)|T(r,s)\rangle^+\,.\end{array} \tag{2.9}$$

In the ' $-$ '-modules, similarly,

$$|S(r,s)\rangle^- = \mathcal{G}_0\ldots\mathcal{G}_{r-1}\cdot|T(r,s)\rangle^- \tag{2.10}$$

The vectors $|S(r,s)\rangle^\pm$ are on level $rs$ over $|\mathsf{h}^\pm(r,s)\rangle_{\rm top}$. The non-trivial part of the construction is finding these $|T(r,s)\rangle^\pm$. The general formulae will be given in (5.3) and (5.7), and a large part of sections 4–7 will be devoted to explaining their meaning.

## 3 Reminder on the MFF construction and related formulae

A motivation for our construction can be taken partly from the Kazama–Suzuki mapping [29], which allows one to map the topological conformal algebra to the universal enveloping of $s\ell(2)_k \oplus [BC]$, where $[BC]$ is an auxiliary fermionic $BC$-system (spin-1 'ghosts'), and $s\ell(2)_k$ is a level-$k$ Kač–Moody algebra:

$$\mathcal{Q} = CJ^+ , \quad \mathcal{G} = \tfrac{1}{k+2} BJ^- , \quad \mathcal{H} = \tfrac{k}{k+2} BC - \tfrac{2}{k+2} J^0 , \tag{3.1}$$

$$\mathcal{T} = \tfrac{1}{k+2}(J^+J^-) - \tfrac{k}{k+2} B\partial C - \tfrac{2}{k+2} BCJ^0 \tag{3.2}$$

Topological central charge $\mathsf{c}$ of the topological generators thus constructed is given by $\mathsf{c} = \frac{3k}{k+2}$.

As ref. [18] indicates, the Kazama–Suzuki mapping induces an isomorphism between the topological and affine $s\ell(2)$ singular vectors[3]: evaluating the image of a topological singular vector $|T(r,s)\rangle^+$ under the Kazama–Suzuki mapping (3.1), (3.2), we find that it literally coincides with an $s\ell(2)$ singular vector:

$$|T(r,s)\rangle^+ \mapsto |\mathrm{MFF}\{r,s\}\rangle^+ \otimes |0\rangle_{BC} \tag{3.3}$$

where $|0\rangle_{BC}$ is the $BC$ vacuum, the vacuum annihilation conditions being $B_{\geq 0}|0\rangle = C_{\geq 1}|0\rangle = 0$, and $|\mathrm{MFF}\{r,s\}\rangle^+$ is a singular vector in the affine $s\ell(2)_k$ Verma module with the highest-weight state $|j_+(r,s)\rangle_{s\ell(2)}$, where

$$j_+(r,s) = \tfrac{r-1}{2} - t\tfrac{s-1}{2} , \qquad r,s \in \mathbb{N} , \qquad t \equiv k+2 . \tag{3.4}$$

The formula (3.3), although it has been thoroughly established only for level $\leq 4$, does nevertheless strongly suggest a 1:1 isomorphism between topological and $s\ell(2)$ singular vectors. A general formula is known for the latter [7] [4]:

$$|\mathrm{MFF}\{r,s\}\rangle = (J_0^-)^{r+(s-1)t}(J_{-1}^+)^{r+(s-2)t}(J_0^-)^{r+(s-3)t}\dots(J_{-1}^+)^{r-(s-2)t}(J_0^-)^{r-(s-1)t}|j_+(r,s)\rangle_{s\ell(2)} . \tag{3.5}$$

These will thus be called the MFF [singular] vectors. The expression (3.5) is in fact used to 'generate' singular vectors by applying analytically continued commutation relations in the universal enveloping algebra of $s\ell(2)_k$:

$$\begin{array}{rcl}
(J_0^-)^p J_m^0 &=& pJ_m^-(J_0^-)^{p-1} + J_m^0(J_0^-)^p , \\
(J_0^-)^p J_m^+ &=& J_m^+(J_0^-)^p + 2pJ_m^0(J_0^-)^{p-1} + p(p-1)J_m^-(J_0^-)^{p-2} , \\
J_m^0(J_{-1}^+)^p &=& p(J_{-1}^+)^{p-1}J_{m-1}^+ + (J_{-1}^+)^p J_m^0 , \\
J_m^-(J_{-1}^+)^p &=& (J_{-1}^+)^p J_m^- + 2p(J_{-1}^+)^{p-1}J_{m-1}^0 + p(p-1)(J_{-1}^+)^{p-2}J_{m-2}^+ - kp\delta_{m-1,0}(J_{-1}^+)^{p-1}
\end{array} \tag{3.6}$$

(which are assumed to hold for arbitrary complex $p$). Upon the repeated use of these relations all the non-positive-integer powers disappear from the MFF formula (3.5), and it thus rewrites in a 'Verma' form, i.e. as an explicitly Verma module element.

It is only this 'Verma' form of $s\ell(2)$ singular vectors, rather than the 'MFF' form (3.5), that can arise from direct computations [18] of topological singular vectors with the help of the Kazama–Suzuki mapping (3.1), (3.2). However, in view of the actual coincidence (3.3) of topological singular vectors with the MFF vectors, an intriguing question is whether the entire MFF construction would have a counterpart for the twisted $N=2$ algebra, in the form of a generating expression for topological singular vectors.

[3] Considered in [18] were only the $s\ell(2)$-singular vectors over the highest-weight state $|j_+(r,s)\rangle$ (see eq. (3.4) below). A similar statement holds for the 'other half' of singular vectors $|\mathrm{MFF}\{r,s\}\rangle^-$, namely those built upon $|j_-(r,s)\rangle$ where $j_-(r,s) = -\frac{1}{2}(r+1) + \frac{1}{2}(k+2)s$ and $r,s \in \mathbb{N}$.

[4] Again, we will consider for brevity only a 'half' of the MFF singular vectors, the $|\ \rangle^+$-ones.

Before proceeding to the topological construction, let us note that the situation described by eqs. (3.5), (3.6) can be viewed as an extension of the universal enveloping algebra of $s\ell(2)_k$ by new elements, *denoted* as $(J^+_{-1})^p$ and $(J^-_0)^p$, and introducing the algebraic rules necessary to deal with the new objects; some of these rules are self-evident due to the 'exponential notation'. Note also that not every commutation relation is defined (for instance, the commutator of $(J^+_{-1})^p$ with $(J^-_0)^{p'}$ is not, unless $p$ or $p'$ is an integer). The same idea works for the topological conformal algebra, although more algebraic rules will have to be written out explicitly.

# 4 Topological algebra intertwiners and generalized Verma modules

## 4.1 Motivation

There are two cases when the 'topological' counterpart of the MFF construction can be built immediately. This happens for a discrete subset of the parameter values, namely when the MFF formula (3.5) does not require any manipulations in order to be transformed into a Verma module element; it can then be rewritten identically in the topological guise, as a formula that describes a subset of topological singular vectors. The two cases are realized either when $s = 1$ or for those $r$ and $s$ and *integral* $t$ for which $|(s-1)t| \leq r$ (so that all exponents in (3.5) are positive integers). In the first case, the $|S(r,1)\rangle^+$ topological singular vectors read

$$\mathcal{Q}_0\mathcal{Q}_1 \ldots \mathcal{Q}_{r-1}\,\mathcal{G}_{-r} \ldots \mathcal{G}_{-2}\mathcal{G}_{-1}\,|\mathsf{h}^+(r,1)\rangle_{\text{top}} \tag{4.1}$$

which under the Kazama–Suzuki mapping becomes just the singular vector $(J^+_{-1})^r|\text{MFF}\{r,1\}\rangle \otimes |0\rangle_{BC}$ (we have thus recovered a representation that has been known for some time [30]). In the second case, a generalization of the previous formula is not difficult to construct, and one finds for the $|T(r,s)\rangle^+$-vector (see (2.8)) the following representation:

$$\begin{aligned}
|T(r,s)\rangle^+ \;=\; & \mathcal{G}_{(\mathsf{h}-s+1)t-1} \ldots \mathcal{G}_{(s-1)t-2}\mathcal{G}_{(s-1)t-1}\,\cdot \\
& \quad \mathcal{Q}_{-(s-1)t} \ldots \mathcal{Q}_{(s-\mathsf{h}-2)t-1}\,\mathcal{Q}_{(s-\mathsf{h}-2)t} \\
& \qquad \ldots \\
& \qquad \mathcal{G}_{(\mathsf{h}-1)t-1} \ldots \mathcal{G}_{t-2}\mathcal{G}_{t-1}\,\cdot \\
& \qquad\quad \mathcal{Q}_{-t} \ldots \mathcal{Q}_{-\mathsf{h}t-1}\,\mathcal{Q}_{-\mathsf{h}t} \\
& \qquad\qquad \mathcal{G}_{\mathsf{h}t-1} \ldots \mathcal{G}_{-2}\mathcal{G}_{-1}\,|\mathsf{h}\rangle_{\text{top}}
\end{aligned} \tag{4.2}$$

where $\mathsf{h} = \mathsf{h}^+(r,s)$ for the appropriate $r$ and $s$; the above conditions on $t$ and $r$ and $s$ guarantee that the difference between the mode numbers of the right and the left factors in each group ('right' − 'left') is a non-negative integer (or $-1$, in which case the product evaluates as 1). This topological singular vector evaluates under the Kazama-Suzuki mapping precisely as $|\text{MFF}\{r,s\}\rangle \otimes |0\rangle_{BC}$, as is easy to check (each group of factors in (4.2) corresponds to some $(J^\pm_{-1,0})^m$ in (3.5)).

## 4.2 Generalized Verma modules

Taking eq. (4.2) as a motivation, one may try to arrive at the general construction by an 'algebraic continuation' of the last formula. One readily sees that the sought formula should involve a continuation with respect to *modes* of the operators. The topological conformal algebra commutators can easily be extended to arbitrary (complex) mode numbers. However, eq. (4.2) also tells us that the desired extension would require objects that represent products of modes of $\mathcal{G}$ and $\mathcal{Q}$ with a *non-integral number of factors*.

This hint proves to be more constructive than it might seem, as the desired objects can in fact be given meaning as intertwining operators (see footnote 1)

$$g(a,b) \quad \text{and} \quad q(a,b) \quad a,b\in\mathbb{C}\,. \tag{4.3}$$

They map between *generalized Verma modules* $\mathcal{V}_{\theta,\mathsf{h}}$, which are related to the ordinary Verma modules by the spectral flow transform [24, 25] (see also [22]). Namely, the spectral flow provides an isomorphism between a generalized Verma module $\mathcal{V}_{\theta,\mathsf{h}}$ (to be defined below) and the ordinary Verma module $V_{\mathsf{h}-\frac{2}{t}\theta}$ from Sec. 2. At the same time, the topological conformal algebra is mapped under the spectral flow into an isomorphic algebra via $\mathcal{A}_n\mapsto\mathcal{A}_n^\theta$, where

$$\begin{aligned}\mathcal{L}_n^\theta &= \mathcal{L}_n+\theta\mathcal{H}_n+\tfrac{\mathsf{c}}{6}(\theta^2+\theta)\delta_{n,0}\,, \qquad \mathcal{H}_n^\theta=\mathcal{H}_n+\tfrac{\mathsf{c}}{3}\theta\delta_{n,0}\,,\\ \mathcal{Q}_n^\theta &= \mathcal{Q}_{n-\theta}\,, \qquad \mathcal{G}_n^\theta=\mathcal{G}_{n+\theta}\end{aligned} \tag{4.4}$$

(which is a 'twisted' form of the corresponding relations from [25]). It is important that the modes of $\mathcal{L}$ and $\mathcal{H}$ remain integral after this transformation. We can then identify the respective $\mathcal{L}$ and $\mathcal{H}$ generators in the different modules. The generalized Verma modules will be considered as modules over an isomorphic image of the topological conformal algebra, in which the fermionic generators are realized as $\mathcal{G}_n^\theta=\mathcal{G}_{\theta+n}$, $\mathcal{Q}_n^\theta=\mathcal{Q}_{-\theta+n}$, while $\mathcal{L}_n$ and $\mathcal{H}_n$ are 'the same' in the different modules:

$$\begin{array}{lcllcllcl}
[\mathcal{L}_m,\mathcal{L}_n] &=& (m-n)\mathcal{L}_{m+n}\,, & [\mathcal{H}_m,\mathcal{H}_n] &=& \tfrac{\mathsf{c}}{3}m\delta_{m+n,0}\,, & & & \\
[\mathcal{L}_m,\mathcal{G}_\nu] &=& (m-\nu)\mathcal{G}_{m+\nu}\,, & [\mathcal{H}_m,\mathcal{G}_\nu] &=& \mathcal{G}_{m+\nu}\,, & m,\ n &\in& \mathbb{Z}\,,\\
[\mathcal{L}_m,\mathcal{Q}_\lambda] &=& -\nu\mathcal{Q}_{m+\lambda}\,, & [\mathcal{H}_m,\mathcal{Q}_\lambda] &=& -\mathcal{Q}_{m+\lambda}\,, & \nu &\in& \theta+\mathbb{Z}\,,\\
[\mathcal{L}_m,\mathcal{H}_n] &=& -n\mathcal{H}_{m+n}+\tfrac{\mathsf{c}}{6}(m^2+m)\delta_{m+n,0}\,, & & & & \lambda &\in& -\theta+\mathbb{Z}\\
\{\mathcal{G}_\nu,\mathcal{Q}_\lambda\} &=& 2\mathcal{L}_{\nu+\lambda}-2\lambda\mathcal{H}_{\nu+\lambda}+\tfrac{\mathsf{c}}{3}(\nu^2+\nu)\delta_{\nu+\lambda,0}\,, & & & & & &
\end{array} \tag{4.5}$$

As can be seen, the commutator of, say, $\mathcal{G}_n^\theta$ and $\mathcal{Q}_m^\theta$ is not the same as that of $\mathcal{G}_n$ and $\mathcal{Q}_m$. However, the difference can be eliminated by a redefinition of the $\mathcal{L}_n$ and $\mathcal{H}_0$ in any given module. We will work in what follows with the 'common' $\mathcal{L}_n$ and $\mathcal{H}_n$ generators.

Then, the highest-weight conditions that determine a generalized highest-weight state can have one of the two forms:

$$\begin{array}{lcll lcll}
\mathcal{L}_m &\approx& 0\,, & m\geq1\,, & \mathcal{Q}_\lambda &\approx& 0\,, & \lambda=-\theta+p\,,\quad p=1,2,\ldots\\
\mathcal{H}_m &\approx& 0\,, & m\geq1\,, & \mathcal{G}_\nu &\approx& 0\,, & \nu=\theta+p\,,\quad p=0,1,2,\ldots
\end{array} \tag{4.6}$$

or

$$\begin{array}{lcll lcll}
\mathcal{L}_m &\approx& 0\,, & m\geq1\,, & \mathcal{Q}_\lambda &\approx& 0\,, & \lambda=-\theta+p\,,\quad p=0,1,2,\ldots\\
\mathcal{H}_m &\approx& 0\,, & m\geq1\,, & \mathcal{G}_\nu &\approx& 0\,, & \nu=\theta+p\,,\quad p=1,2,\ldots
\end{array} \tag{4.7}$$

However, these actually determine a class of modules (see [36]) more general than those that we are going to consider in this paper. The generalized *topological* highest-weight conditions that we will work with are obtained by strengthening each of the previous ones:

$$\begin{array}{lcll lcll}
\mathcal{L}_m|\theta,\mathsf{h}\rangle_{\rm top} &=& 0\,, & m\geq1\,, & \mathcal{Q}_\lambda|\theta,\mathsf{h}\rangle_{\rm top} &=& 0\,, & \lambda=-\theta+p\,,\quad p=0,1,2,\ldots\\
\mathcal{H}_m|\theta,\mathsf{h}\rangle_{\rm top} &=& 0\,, & m\geq1\,, & \mathcal{G}_\nu|\theta,\mathsf{h}\rangle_{\rm top} &=& 0\,, & \nu=\theta+p\,,\quad p=0,1,2,\ldots
\end{array} \tag{4.8}$$

The corresponding states $|\theta,\mathsf{h}\rangle_{\rm top}$ will be called the (generalized) topological highest-weight states. Then the generalized Verma modules $\mathcal{V}_{\theta,\mathsf{h}}$ are freely generated from $|\theta,\mathsf{h}\rangle_{\rm top}$ by those of the generators

from (4.5) that are not declared annihilation operators in (4.8) [5]. The eigenvalues of the generators $\mathcal{L}_0$ and $\mathcal{H}_0$ from (4.5) on a topological highest-weight state in $\mathcal{V}_{\theta,\mathsf{h}}$ are given by

$$\begin{aligned} \mathcal{H}_0 + \theta &\approx \mathsf{h} \\ \mathcal{L}_0 + \theta\mathcal{H}_0 + \tfrac{\mathsf{c}}{6}(\theta^2 + \theta) &\approx 0 \end{aligned} \tag{4.9}$$

The ordinary topological Verma modules, where $\mathcal{L}_0|\mathsf{h}\rangle_{\mathrm{top}} = 0$ and $\mathcal{H}_0|\mathsf{h}\rangle_{\mathrm{top}} = \mathsf{h}|\mathsf{h}\rangle_{\mathrm{top}}$, are thus a particular case of a more general situation described by (4.8) and (4.9).

To see which ordinary Verma module is isomorphic to $\mathcal{V}_{\theta,\mathsf{h}}$, we have to use the basis in which the commutation relations of the algebra become identical to (2.1); for $\mathcal{H}_0$ this amounts to accounting for the $\theta\frac{\mathsf{c}}{3}$ term in (4.4). We then see that

$$\mathcal{V}_{\theta,\mathsf{h}} \sim V_{\mathsf{h}-\frac{2}{t}\theta}\,, \tag{4.10}$$

as claimed above.

### 4.3 Algebra of the intertwiners

By definition, the mapping of the intertwiners (4.3) preserves the highest-weight conditions (4.6) *or* (4.7) when these operators are applied as:

$$g(\theta', \theta - 1) \,:\, \mathcal{V}_{\mathsf{h},\theta} \to \mathcal{V}_{\mathsf{h},\theta'}\,, \qquad q(-\theta', -\theta - 1) \,:\, \mathcal{V}_{\mathsf{h},\theta} \to \mathcal{V}_{\mathsf{h},\theta'}\,, \tag{4.11}$$

which thus relates the arguments of $g$ and $q$ to the $\theta$-parameters of the corresponding generalized Verma modules. We will further require that the *'topological'* highest-weight conditions (4.8) be preserved under the mappings, as this will be needed in the construction of singular vectors. As we will see, when the intertwiners (4.3) map between the generalized topological Verma modules, the highest weight states map as follows. $g(\theta', \theta - 1)$ and $q(-\theta', -\theta - 1)$ take the topological highest-weight state in $\mathcal{V}_{\theta,\mathsf{h}}$ into the topological highest-weight state in $\mathcal{V}_{\theta',\mathsf{h}}$ (i.e., the conditions (4.8) are preserved) if and only if

$$\theta' = \mathsf{h}t - \theta - 1 \qquad \text{or} \qquad \theta' = (\mathsf{h}+1)t - \theta - 1 \tag{4.12}$$

respectively. A second application of an intertwiner of the same sort maps back to the original module, while the other intertwiner maps to a new module:

$$\begin{tikzcd} {} \arrow[r, bend left, "g"] & \bullet \arrow[l, bend left, "g"] \arrow[r, bend left, "q"] & \bullet \arrow[l, bend left, "q"] \arrow[r, bend left, "g"] \arrow[d, phantom, "\mathcal{V}_{\theta,\mathsf{h}}"] & \bullet \arrow[l, bend left, "g"] \arrow[r, bend left, "q"] & {} \arrow[l, bend left, "q"] \\ & & {} & & \end{tikzcd} \tag{4.13}$$

In order to encounter an ordinary Verma module $\mathcal{V}_{0,\mathsf{h}} \equiv V_{\mathsf{h}}$ among the modules thus obtained from $\mathcal{V}_{\theta,\mathsf{h}}$, the values of $\theta$ should therefore be $\theta_i$, $i = 0, 1, 2, \ldots$, where

$$\theta_i = \begin{cases} (\mathsf{h} - j)t - 1 & i = 2j + 1\,, \\ jt & i = 2j\,. \end{cases} \tag{4.14}$$

Heuristically, the intertwiners $g(a, b)$ or $q(a, b)$ are a continuation of the product of modes $\mathcal{G}_a \ldots \mathcal{G}_b$ and $\mathcal{Q}_a \ldots \mathcal{Q}_b$ respectively to arbitrary (complex) $a$ and $b$. This is formalized as the following property:

[5] We think of the topological central charge $\mathsf{c} \equiv 3(t-2)/t$ as fixed; a more rigorous notation for the modules should be $\mathcal{V}_{\theta,\mathsf{h},t}$.

Whenever the *length* $\ell = b - a + 1$ of an intertwiner $g(a,b)$ (or $q(a,b)$) becomes a positive integer, that operator rewrites as the corresponding product of modes

$$g(a, a+n) = \prod_{i=0}^{n} \mathcal{G}_{a+i}\,, \quad q(a, a+n) = \prod_{i=0}^{n} \mathcal{Q}_{a+i}\,, \quad n = -1, 0, 1, 2, \ldots \tag{4.15}$$

(in the case $n = -1$ the product evaluates as 1). As long as the intertwiners preserve the topological highest-weight conditions, the intertwiners of positive integral length, once they send the highest-weight state into a state *in the same* module, are in fact singular vectors. On the other hand, $g(a, a-n)$ for a positive integral $n$ does *not* evaluate as 1 unless $n = 1$. The intertwiners $g(a, a-n)$ of negative integral length $-n+1$ represent *co*singular vectors.

The intertwiners inherit a number of further properties from the products of modes. These properties will be imposed, in addition to (4.15), as a consistent set of commutation relations and certain rearrangement rules for $q(a,b)$ and $g(a,b)$ with arbitrary complex $a$ and $b$. We will formulate these properties for the $g$ intertwiners, the '$q$'-case being completely analogous. To begin with, these are the relations

$$g(a, b-1)\, g(b, c) = g(a, c)\,, \qquad g(a+1, a) = 1\,. \tag{4.16}$$

It follows then that, in particular,

$$\mathcal{G}_a\, g(a+1, b) = g(a, b)\,, \quad g(a, b)\, \mathcal{G}_{b+1} = g(a, b+1)\,. \tag{4.17}$$

Further,

$$\mathcal{G}_a\, g(b, c) = 0\,, \qquad a - b = 0, 1, 2, \ldots\,, \quad c - b + 1 \neq 0, 1, 2, \ldots\,, \tag{4.18}$$

and, similarly,

$$g(a, b)\, \mathcal{G}_c = 0\,, \qquad b - c = 0, 1, 2 \ldots\,, \quad b - a + 1 \neq 0, 1, 2, \ldots\,. \tag{4.19}$$

(in the case of a positive integral length $c-b+1$ or $b-a+1$, the reduction (4.15) has to performed first, after which the vanishing conditions become the usual ones in the universal enveloping of the $N=2$ algebra).

Relations similar to (4.16)–(4.19) hold for the $q$ intertwiners (provided $\mathcal{G}$ is replaced with $\mathcal{Q}$). Eqs. (4.18), (4.19) can be viewed as a manifestation of the Pauli principle, namely that $g(a,b)$ and $q(a,b)$ represent states from $a$ to $b$ filled with the corresponding fermions. The relations (4.16) and (4.17), as well as those that we are going to introduce, can be checked to hold when the intertwiners involved reduce to a product of the corresponding modes (it should be clear, however, that the inverse is far from being true: not every relation that holds in the universal enveloping of (2.1) extends to the intertwiners!).

To see how the 'Pauli principle' allows us to arrive at the necessary commutation relations, consider for instance commuting the mode $\mathcal{H}_p$, with $p$ an integer $\geq 1$, through $g(a,b)$. In the universal enveloping algebra, with $a$ and $b$ positive integers such that $b > a$, we have from (2.1)

$$[\mathcal{H}_p,\, g(a,b)] \equiv [\mathcal{H}_p,\, \prod_{i=a}^{b} \mathcal{G}_i] = \sum_{j=a}^{b} \Bigl(\prod_{i=a}^{j-1} \mathcal{G}_i\Bigr)\, \mathcal{G}_{j+p} \Bigl(\prod_{i=j+1}^{b} \mathcal{G}_i\Bigr) \tag{4.20}$$

However, the terms in the sum vanish whenever $j + p$ (which so far is an integer) is inside the segment $[j+1, b]$. Therefore, the RHS of the last formula contains only the terms with $j = b - p + 1, \ldots, b$, hence

$$\mathcal{H}_p\, g(a,b) = g(a,b)\, \mathcal{H}_p + \sum_{j=1}^{p} g(a, b-p+j-1)\, \mathcal{G}_{b+j} \prod_{i=1}^{p-j} \mathcal{G}_{b-p+j+i}\,, \qquad p \geq 1\,. \tag{4.21}$$

When written in this form, the formula can be used for *arbitrary* $a$ and $b$, since the RHS contains an integral number of terms and the product involves an integral number of factors, independently of the values of $a$ and $b$. It is in this form that we fix the commutation relation $[\mathcal{H}_p,\, g(a,b)]$. For $\mathcal{H}_0$, the definition is motivated by even simpler relations

$$[\mathcal{H}_0,\,\prod_{i=m}^{n}\mathcal{G}_i] = (n-m+1)\prod_{i=m}^{n}\mathcal{G}_i\,,\qquad [\mathcal{H}_0,\,\prod_{i=m}^{n}\mathcal{Q}_i] = (-n+m-1)\prod_{i=m}^{n}\mathcal{Q}_i \tag{4.22}$$

which in an obvious way rewrite in terms of $g(m,n)$ and $q(m,n)$ and then extend to arbitrary $m$ and $n$. The formulae that describe commutators of $\mathcal{L}_{\geq 0}$ with $g(a,b)$ (and $q(a,b)$), can be derived similarly to (4.21), with use being made of the commutators $[\mathcal{L}_n,\,\mathcal{G}_m]$ and $[\mathcal{L}_n,\,\mathcal{Q}_m]$ from (2.1):

$$[\mathcal{L}_0,\,g(a,b)] = -\tfrac{1}{2}(a+b)(b-a+1)\,g(a,b)\,,\qquad [\mathcal{L}_0,\,q(a,b)] = -\tfrac{1}{2}(a+b)(b-a+1)\,q(a,b) \tag{4.23}$$

and

$$\mathcal{L}_p g(a,b) = g(a,b)\mathcal{L}_p + \sum_{l=0}^{p-1} g(a,b-l-1)\,[\mathcal{L}_p\,,\,\mathcal{G}_{b-l}]\;\mathcal{G}_{b-l+1}\ldots\mathcal{G}_b\,,\quad p\geq 1\,. \tag{4.24}$$

Finally, when checking the highest-weight conditions, one also uses the following consequence of the Pauli principle:

$$\mathcal{G}_{\theta+n}\,g(\theta,\theta'-1)\,q(-\theta',-\theta''-1)\,\ldots\,g(\widetilde{\theta},-1)\,|\mathsf{h}\rangle_{\rm top} = 0\,,\qquad n=1,2,\ldots \tag{4.25}$$

In addition, one can evaluate $\mathcal{Q}_{-\theta}\,g(\theta,\theta'-1)\,q(-\theta',-\theta''-1)\,\ldots\,g(\widetilde{\theta},-1)\,|\mathsf{h}\rangle_{\rm top}$ by writing the leftmost intertwiner as $\mathcal{G}_\theta\,g(\theta+1,\theta'-1)$, whence

$$\begin{aligned}&\mathcal{Q}_{-\theta}\,g(\theta,\theta'-1)\,q(-\theta',-\theta''-1)\,\ldots\,g(\widetilde{\theta},-1)\,|\mathsf{h}\rangle_{\rm top} =\\ &\qquad\{\mathcal{Q}_{-\theta},\,\mathcal{G}_\theta\}\,g(\theta+1,\theta'-1)\,q(-\theta',-\theta''-1)\,\ldots\,g(\widetilde{\theta},-1)\,|\mathsf{h}\rangle_{\rm top}\end{aligned} \tag{4.26}$$

which holds provided $\theta$, $\theta'$, …, $\widetilde{\theta}$ match as in (4.14). Here, the commutator $\{\mathcal{Q}_{-\theta},\,\mathcal{G}_\theta\}$ should be taken from (4.5) and the resulting modes of $\mathcal{L}$ and $\mathcal{H}$ then evaluated as explained above ('symmetric' formulae hold for $q\leftrightarrow g$). This then allows us to check whether or not the topological highest-weight conditions (4.8) are preserved under the action of $g$ and $q$, and this is how eqs. (4.12) have been obtained.

Further, using the equations (4.21) and (4.24), we conclude from (4.26) that

$$\mathcal{Q}_{-\theta+n}\,g(\theta,\theta'-1)\,q(-\theta',-\theta''-1)\,\ldots\,g(\widetilde{\theta},-1)\,|\mathsf{h}\rangle_{\rm top} = 0\,,\quad n=1,2\ldots \tag{4.27}$$

which may be interpreted again in the spirit of the Pauli principle: all the 'inner' commutators in (4.27) obtained while plugging $\mathcal{Q}_{-\theta+n}$ on the right will necessarily produce terms that vanish either due to the previously established highest-weight conditions, or due to the Pauli principle.

For negative-moded $\mathcal{H}$ and $\mathcal{L}$, the following formulae are derived by a simple modification of the argument (4.20)–(4.21):

$$\begin{aligned} g(a,b)\mathcal{L}_p &= \mathcal{L}_p g(a,b) + \sum_{l=0}^{-p-1}\mathcal{G}_a\ldots\mathcal{G}_{a+l-1}\,[\mathcal{G}_{a+l}\,,\,\mathcal{L}_p]\;g(a+l+1,b)\,,\quad p\leq -1\\ g(a,b)\mathcal{H}_p &= \mathcal{H}_p g(a,b) + \sum_{l=0}^{-p-1}\mathcal{G}_a\ldots\mathcal{G}_{a+l-1}\,[\mathcal{G}_{a+l}\,,\,\mathcal{H}_p]\;g(a+l+1,b)\,,\quad p\leq -1\,.\end{aligned} \tag{4.28}$$

These relations will in fact be the basic tool in what follows (their analogues with $g$ replaced by $q$ are straightforward to formulate). As we will see, eqs. (4.28) are a 'topological' counterpart of the $s\ell(2)$ formulae (3.6). However, while eqs. (3.6) allow one to commute the powers of $J^+_{-1}$ and $J^-_0$ through any of the generators $J^{\pm,0}_m$, in the topological case we will never need the analogue of (4.28) with $\mathcal{L}_p$, $p=-1,-2,\ldots$, replaced with $\mathcal{Q}_{-\theta+p}$ or $\mathcal{G}_{\theta+p}$. The mechanism that spares us commuting the fermions through the intertwiners (the appearance of an '$\mathcal{L}$-$\mathcal{H}$-skeleton') will be explained below.

# 5 Topological singular vectors and generalized Verma modules

The question of whether a module admits a singular vector can be reformulated in terms of the existence of an intertwiner from that module to itself. The mappings implemented by the intertwiners have been tuned (see (4.14)) to preserve the topological highest-weight conditions (4.8). Thus, moving along the arrows in the diagram (4.13), we are interested in the case when a sequence of oscillating $q$- and $g$- arrows makes a loop:

$$\begin{tikzcd} & & & \mathcal{V}_{\theta,\mathsf{h}} \arrow[dr, "g"] & & & \\ & & \bullet \arrow[ur, "g"] & & \bullet \arrow[dr, "q"] & & \\ & \bullet \arrow[ur, "q"] & & & & \bullet \arrow[dr] & \\ \vdots \arrow[ur] & & & & & & \vdots \end{tikzcd} \tag{5.1}$$

Then, $\mathcal{V}_{\theta,\mathsf{h}}$ will have a singular vector. In the following, we will apply this strategy to constructing the topological singular vectors in the usual Verma modules, which will mean taking $\mathcal{V}_{\theta,\mathsf{h}}$ in (5.1) to be $\mathcal{V}_{0,\mathsf{h}} \equiv V_{\mathsf{h}}$, the module built over $|\mathsf{h}\rangle_{\rm top}$. Thus, after $2\ell+1$ steps along the arrows in (5.1), we arrive at a state (see (4.14))

$$\begin{aligned} &|\mathsf{h},t,\ell\rangle^{+}_{\rm top} = \\ &\quad g((\mathsf{h}-\ell)t-1,\ell t-1)\,q(-\ell t,(\ell-1-\mathsf{h})t)\ \ldots \\ &\qquad g((\mathsf{h}-1)t-1,t-1)\,q(-t,-\mathsf{h}t)\,g(\mathsf{h}t-1,-1)\,|\mathsf{h}\rangle_{\rm top} \end{aligned} \tag{5.2}$$

The condition for this to belong to $V_{\mathsf{h}}$ and be non-vanishing is that $(\mathsf{h}-\ell)t-1$ be a negative integer, say $-r$. We thus recover the parametrization for $\mathsf{h}=\mathsf{h}^{+}$ as in (2.7) in terms of two positive integers $r$ and $s=\ell+1$ (the formula for $\mathsf{h}^{-}$ is recovered similarly, by considering a loop starting and ending with a $q$ intertwiner). From now on, we will denote $V_{\mathsf{h}^{\pm}(r,s)}$ as $V^{\pm}_{r,s}$.

By writing the state (5.2) for a closed loop, we arrive at the formula for topological singular vectors in the topological Verma module $V^{+}_{r,s}$:

$$\begin{aligned} |T(r,s)\rangle^{+} \;=\;& g(-r,(s-1)t-1)\,q(-(s-1)t,r-1-t)\ \ldots \\ & g((s-2)t-r,t-1)\,q(-t,r-1-t(s-1))\,g((s-1)t-r,-1)\,|\mathsf{h}^{+}(r,s)\rangle_{\rm top}\,, \\ & r,s\in\mathbb{N} \end{aligned} \tag{5.3}$$

Every 'intermediate' module appears in (5.1) twice, which guarantees that we end up with a singular vector in the original module[6]: denoting the intermediate modules as $\mathcal{V}^{\pm}_{r,s}$, we have

$$\begin{array}{c} V^{+}_{r,s}\supset\mathcal{V}^{-,\theta_{2s-1}}_{r,s}\leftarrow\mathcal{V}^{+,\theta_{2s-2}}_{r,s-1}\ldots\leftarrow\mathcal{V}^{\mp,\theta_s}_{r,1}\leftarrow\mathcal{V}^{\pm,\theta_{s-1}}_{r,1}\leftarrow\ldots\leftarrow\mathcal{V}^{-,\theta_1}_{r,s-1}\leftarrow V^{+}_{r,s}\,, \\ \cap \\ \mathcal{V}^{\pm,\theta_{s-1}}_{r,1} \end{array} \tag{5.4}$$

(with $\mathcal{V}^{\mp,\theta_i}_{r,s}$ standing for the image of $\mathcal{V}^{\mp}_{r,s}$ under the spectral flow transformation with the $\theta_i$ parameter taken from (4.14)). In the 'centre' of (5.4), we have the mapping given by the intertwiner $g(a,b)$ (for $s$ odd) or $q(a,b)$ (for $s$ even) of positive integral length $b-a+1=r$. Therefore, while moving from the right-hand end to the centre of (5.4), the application of the intertwining operators decreases the value of $s$ and leads to a module in which $s=1$ and hence the singular vectors have the simplest form (4.1). It is this $|T(r,1)\rangle$ vector in the appropriate generalized Verma module, given by the product of $r$ modes

[6]and also that 'intermediate' singular vectors $|T(r,s'<s)\rangle^{\pm}$ follow from (5.3), as we will see in section 7.

of $\mathcal{G}$ (or $\mathcal{Q}$), that we encounter in the 'centre' of the formula (5.3). The left 'half' of the diagram (5.4) does bring us back into the original module.

Note also that the topological $U(1)$ charge (the eigenvalue of $\mathcal{H}_0$ on the generalized highest-weight state) is mapped under (5.4) as

$$\ldots \leftarrow h-4 \leftarrow \tfrac{2}{t}-h+2 \leftarrow h-2 \leftarrow \tfrac{2}{t}-h \leftarrow h \tag{5.5}$$

which suggests an underlying structure similar to the affine Weyl group one has in the $s\ell(2)$ case.

Analogous formulae, with $g \leftrightarrow q$, and $\mathcal{Q} \leftrightarrow \mathcal{G}$ and $+ \leftrightarrow -$, exists for $|T(r,s)\rangle^-$: the counterpart of (5.2) reads

$$\begin{aligned} &q(-(\mathsf{h}+\ell+1)t+1, \ell t-1)\, g(-\ell t, (\mathsf{h}+\ell)t-2)\ \ldots \\ &\quad q(1-2t-\mathsf{h}t, -1+t)\, g(-t, -2+t+\mathsf{h}t)\, q(1-t-\mathsf{h}t, -1)\, |\mathsf{h}\rangle_{\rm top}\,, \end{aligned} \tag{5.6}$$

whence

$$\begin{aligned} |T(r,s)\rangle^- \;=\; & q(-r, -1-t+st)\, g(t-st, -1+r-t)\ \ldots \\ & q(-r-2t+st, -1+t)\, g(-t, -1+r+t-st)\, q(-r-t+st, -1)\, |\mathsf{h}^-(r,s)\rangle_{\rm top} \end{aligned} \tag{5.7}$$

The action of the intertwiners does preserve the topological highest-weight conditions (4.8) at every step, therefore these conditions hold for the states (5.3) and (5.7). In the next section we will describe an operational scheme allowing us to rewrite any state of the form (5.3) and (5.7) as a Verma module element, i.e. as a polynomial in $\mathcal{L}_{m<0}$, $\mathcal{H}_{m<0}$, $\mathcal{G}_{m<0}$ and $\mathcal{Q}_{m<0}$ ($m$ integral) acting on $|\mathsf{h}^{\pm}(r,s)\rangle_{\rm top}$, with no intertwiners involved. This will amount to a rearrangement of (5.3), (5.7) by a systematic use of the properties of the intertwiners from section 4.

# 6 Back to the Verma modules

We have seen that the general construction for topological singular vectors satisfies the required highest-weight conditions. In this section, we will transform (5.3) into an explicitly 'Verma' form. (The other half of the topological singular vectors, eqs. (5.7), are treated similarly).

Let us consider for definiteness the case when the $r$ modes in the centre of (5.3) are $\mathcal{Q}$ modes: $q(-\frac{st}{2}, -\frac{st}{2}+r-1) = \mathcal{Q}_{-\frac{st}{2}} \ldots \mathcal{Q}_{-\frac{st}{2}+r-1}$. As follows from (4.14) and (5.3), the lengths of the intertwiners $g(a,b)\cdot\ldots\cdot g(c,d)$ that occupy symmetric positions with respect to the centre of the formula (5.3) always add up to a positive integer, namely to $2r$: $(d-c+1)+(b-a+1)=2r$ (the same is true for any 'symmetric' pair $q(a,b)\ldots q(c,d)$ as well). The aim of the manipulations that follow is to make these intertwiners meet in such a way that the formula (4.16) would apply, which would then allow us to replace the resulting intertwiner of integral length with a product of modes.

By repeatedly (precisely $r$ times) applying to $g(a,b)$ the identical transformation from eq. (4.17),

$$g(a,b) \to g(a,b-1)\mathcal{G}_b\,, \tag{6.1}$$

we bring it to the form of $g(a,c-1)$ times a product of modes of $\mathcal{G}$. Thus, in the case we have chosen for definiteness, the centre of the formula (5.3) rewrites as

$$\begin{aligned} & g(-r+\tfrac{st}{2}-t, \tfrac{st}{2}-1)\, q(-\tfrac{st}{2}, -\tfrac{st}{2}+r-1)\, g(\tfrac{st}{2}-r, \tfrac{st}{2}-t-1) \\ & = g(\tfrac{st}{2}-t-r, \tfrac{st}{2}-r-1) \\ & \quad \cdot \underbrace{\mathcal{G}_{\frac{st}{2}-r}\ \ldots\ \mathcal{G}_{\frac{st}{2}-1}}\ \mathcal{Q}_{-\frac{st}{2}}\ \ldots\ \mathcal{Q}_{-\frac{st}{2}+r-1} g(\tfrac{st}{2}-r, \tfrac{st}{2}-t-1) \end{aligned} \tag{6.2}$$

According to (4.8), the underbraced operators are annihilators with respect to the generalized highest-weight state $g(\frac{st}{2}-r,\frac{st}{2}-t-1)\ldots|\mathsf{h}^{+}(r,s)\rangle_{\rm top}$ that we have on the right of the modes. This is a general fact, as can be seen from the structure of (5.3) and the 'highest-weight' conditions (4.8): rewriting a 'symmetric' pair of intertwiners $g(a,b)\,\ldots\,g(c,d)$ as $g(a,c-1)(\prod\mathcal{G})\,\ldots\,g(c,d)$, we always have in $\prod\mathcal{G}$ precisely $r$ modes that annihilate the state $g(c,d)\ldots|\mathsf{h}\rangle$. All these annihilators have to be commuted to the right and killed on the generalized highest-weight state. Generically, more annihilation operators would appear from the commutators, and these, too, have to be moved right. Any $\mathcal{L}_{\geq 0}$ or $\mathcal{H}_{\geq 0}$ that appears in the course of commutation should also be evaluated on the vacuum, according to (4.8) and (4.9).

The next step is to observe that all the operators $\mathcal{G}$ that might be present between the intertwiners after all the highest-weight conditions have been used, are such that the rule (4.18) applies. This is again true in general, because the modes in question have been split off from the left-hand intertwiner, and then commuted through the other operators which necessarily were creators, hence the mode numbers could only decrease and therefore the condition required in (4.18) fulfills.

Therefore, in the '$\mathcal{G}$'-case under consideration, we can apply the rules (4.18) after having commuted all the $\mathcal{G}$'s to the left. Practically, it suffices to commute them as far left as (and through) the leftmost $\mathcal{Q}$ operator. We will then be left with a polynomial in modes of *only* $\mathcal{L}$ and $\mathcal{H}$.

This is a crucial circumstance because the $\mathcal{L}$ and $\mathcal{H}$ modes can be commuted through the left intertwiner according to the formulae (4.28). When this is done, there will be no more modes between the two $g$ intertwiners, and the formula (4.16) will apply. The resulting intertwiner $g(\mu,\nu)$ will *always* have an integral length $\lambda=\nu-\mu+1$, as can be seen from (5.3). Whenever $\lambda$ is non-negative, the intertwiner gets replaced by a product of modes,

$$g(\mu,\nu)=\prod_{i=0}^{\lambda-1}\mathcal{G}_{\mu+i}\,,\quad\text{iff}\quad\lambda\equiv\nu-\mu+1=0,1,2,\ldots\tag{6.3}$$

and the innermost $g$ intertwiners have therefore been completely eliminated, resulting in a sum of terms that contain only the modes of the topological algebra generators. In each of these terms, one can now apply the same strategy to the embracing $q$-intertwiners, and so on by induction. However, it follows from (4.28) that when $\mathcal{L}_n$ or $\mathcal{H}_n$ is carried through $g(a,b)$, it gives rise to a $g$ intertwiner of a smaller length, and this is the origin of negative-length intertwiners. In some terms we would thus have $g(\mu,\nu)$ of a negative integral length instead of (6.3). It thus remains to give meaning to the remaining intertwiners of negative integral length.

Here we can be guided by associativity in the universal enveloping algebra and use it together with the generalized highest-weight and eigenvalue conditions given above. Then the negative-length intertwiners are 'resolved' as follows. Consider the state

$$\begin{aligned}g(\mu-1,\theta_{n-1}-1)\,q(-\theta_{n-1},-\theta_{n-2}-1)\,\ldots\,g(\theta_1,-1)\,|\mathsf{h}\rangle_{\rm top}&=\\ \mathcal{G}_{\mu-1}g(\mu,\theta_{n-1}-1)\,q(-\theta_{n-1},-\theta_{n-2}-1)\,\ldots\,g(\theta_1,-1)\,|\mathsf{h}\rangle_{\rm top}&\end{aligned}\tag{6.4}$$

and evaluate on it the operator $\mathcal{Q}_{-\mu+1}$ by first commuting it with $\mathcal{G}_{\mu-1}$ as $\{\mathcal{Q}_{-\mu+1},\,\mathcal{G}_{\mu-1}\}=\frac{\mathsf{c}}{3}(-1+\mu)\mu-2(1-\mu)\mathcal{H}_0+2\mathcal{L}_0$ and then exploiting the highest-weight conditions (4.9). We thus arrive at the relation

$$g(\mu,\theta_{n-1}-1)\ldots\,g(\theta_1,-1)\,|\mathsf{h}\rangle_{\rm top}=\frac{1}{\Lambda(\mu,\theta_{n-1},\mathsf{h})}\,\mathcal{Q}_{1-\mu}\,g(\mu-1,\theta_{n-1}-1)\ldots\,g(\theta_1,-1)\,|\mathsf{h}\rangle_{\rm top}\tag{6.5}$$

where $\Lambda$ is the eigenvalue of $\frac{\mathsf{c}}{3}(-1+\mu)\mu-2(1-\mu)\mathcal{H}_0+2\mathcal{L}_0$ on the LHS of (6.5),

$$\Lambda(\theta+N,\theta,\mathsf{h})=(N-1)\left(2\tfrac{\mathsf{c}}{3}\theta+\tfrac{\mathsf{c}}{3}N+2\mathsf{h}-2\theta\right)\,.\tag{6.6}$$

The result of applying (6.5) to intertwiners of negative length is that the length increases by one, $g(\mu, *) \to g(\mu-1, *)$. The rearrangement (6.5) therefore applies successively to the states

$$(modes)\, g(\theta_{n-1}+N, \theta_{n-1}-1)\, q(-\theta_{n-1}, -\theta_{n-2}-1)\, \ldots\, g(\theta_1, -1)\, |\mathsf{h}\rangle_{\rm top} \tag{6.7}$$

starting with the higher $N$, down to $N=2$. The $\mathcal{G}$-operators that can be present among the *modes*, are of the form $\mathcal{G}_{\theta_{n-1}+p}$ with integral $p$. If, in particular, $\mathcal{G}_{\theta_{n-1}+N-2}$ is encountered, then the rule (4.17) should be applied, which would increase the length of the remaining intertwiner even further.

The rule (6.5) is thus used repeatedly, allowing us to evaluate $g(\theta_{n-1}+m, \theta_{n-1}-1)\ldots\, |\mathsf{h}\rangle_{\rm top}$ in terms of $g(\theta_{n-1}+m-1, \theta_{n-1}-1)\ldots\, |\mathsf{h}\rangle_{\rm top}$ for $m \geq 2$. A crucial fact is that after all these rearrangements have been applied in the course of evaluation of a singular vector, *all* the remaining states

$$(modes)\cdot g(\theta_{n-1}+1, \theta_{n-1}-1)\, q(-\theta_{n-1}, -\theta_{n-2}-1)\, \ldots g(\theta_1, -1)\, |\mathsf{h}\rangle_{\rm top} \tag{6.8}$$

cancel! This cancellation is a non-trivial property of the whole scheme; a related fact is that using the formula (6.5) always preserves the coefficients in $t^{-M}\cdot\mathbb{Q}[t]$ (polynomials with rational coefficients times a certain negative power of $t$ that may come out of the highest-weights (2.7)), i.e. no rational dependence on $t$ arises, apart from a possible $t^{-M}$.

The rule (6.5) gives the prescription to deal with intertwiners of negative integral lengths when evaluating the singular vectors. We see that a $g$ intertwiner of a negative integral length $L$, which heuristically stands for the 'absence of $(-L)$ $\mathcal{G}$-modes', does basically reduce on a generalized highest weight state to an insertion of a product of modes of the other fermion, $\mathcal{Q}$, times an overall normalization (more terms come out of the algebra (4.5) when using (6.5) repeatedly). Such a rearrangement would not work for arbitrary expressions with negative-length intertwiners, since the latter do in general take us outside a given Verma module. Consider, for example, $|X\rangle = g(1,-1)|\mathsf{h}\rangle_{\rm top}$, which is formally a cosingular vector with respect to $|\mathsf{h}\rangle_{\rm top}$: $|\mathsf{h}\rangle_{\rm top} = \mathcal{G}_0|X\rangle$, Obviously, it would generically not be in the Verma module built upon $|\mathsf{h}\rangle_{\rm top}$.

We see that the pair of intertwiners $g(a,b)\cdot(\text{modes})\cdot g(c,d)$ in (5.3) or (5.7) do always 'decay' into modes of topological conformal algebra generators, and thus the formula for topological singular vector rewrites as a sum of several terms, in each of which the number of intertwiners has decreased by two.

The procedure of 'eliminating' the intertwining operators symmetric with respect to the centre of (5.3) then repeats mutatis mutandis (with $q \leftrightarrow g$, $\mathcal{Q} \leftrightarrow \mathcal{G}$, etc.). The rules formulated above (together with their '$q$-mirror') are sufficient to carry the evaluation to the end. Eventually, we will be left with a singular vector in the *Verma* module. The expressions obtained in the course of this evaluation acquire more and more complicated structure as a result of commuting the fermionic operators $\mathcal{Q}$ or $\mathcal{G}$ first to the generalized highest-weight state and then, for the remaining modes, to the intertwiner standing on the left of the modes.

Therefore, the expression (5.3) has been given the meaning of a general algebraic construction for topological singular vectors in Verma modules.

# 7 Topological singular vectors via recursion relations

As noted above, a very useful reformulation of the procedure described in the last section can be given in terms of recursion relations that allow one to construct a singular vector $|S(r,s)\rangle^{\pm}$ out of $|S(r,s-1)\rangle^{\mp}$.

Observe that the above step-by-step evaluation of singular vector $|S(r,s)\rangle^{+}$ according to eq. (5.3) has a matrioshka structure: 'inside' the vector $|S(r,s)\rangle^{+}$ there sit 'smaller' singular vectors $|S(r,s')\rangle^{\pm}$ with $1 \leq s' \leq s$ and $+$ or $-$ oscillating from one step to another; in the centre, as noted in the previous

section, there is the $r1$ singular vector. The 'Verma' forms of these 'intermediate' singular vectors are arrived at as soon as a given pair of intertwiners $g(a,b)\,\dots\,g(c,d)$ or $q(a,b)\,\dots\,q(c,d)$ is 'eliminated' (one actually applies also the (inverse) spectral flow transform (4.4) to the modes of $\mathcal{L}$, $\mathcal{H}$, $\mathcal{G}$ and $\mathcal{Q}$, sandwiched between the remaining intertwiners).

Since each 'intermediate' expression is therefore a singular vector, we can simply take an *arbitrary* singular vector (not necessarily an $r1$) as a starting point for producing the higher ones (those with bigger $s$). Then the scheme we applied in Section 6 reformulates as follows.

Assume for definiteness that we start with a '$-$'-vector $|S(r,s-1)\rangle^-$ in the Verma module $V^-_{r,s-1}$ over $|\mathsf{h}\rangle_{\rm top} \equiv |\mathsf{h}^-(r,s-1)\rangle_{\rm top}$; let us then construct $|S(r,s)\rangle^+$ out of it. Taking $|S(r,s-1)\rangle^-$, it will be assumed that it is written in the form when all the annihilation operators have been evaluated on the highest-weight state, therefore the singular vector is given by a polynomial in creation operators acting on the appropriate highest-weight state. Then the following sequence of steps applies (two examples are given in the Appendix).

Step 1. Apply to $|S(r,s-1)\rangle^-$ the spectral flow transformation

$$\begin{aligned} \mathcal{Q}_n &\mapsto \mathcal{Q}_{n-\theta}\,, &\qquad \mathcal{L}_n &\mapsto \mathcal{L}_n + \theta\mathcal{H}_n\,,\\ \mathcal{G}_n &\mapsto \mathcal{G}_{n+\theta}\,, &\qquad \mathcal{H}_n &\mapsto \mathcal{H}_n\,, \end{aligned} \tag{7.1}$$

with

$$\theta = (s-1)t - r \tag{7.2}$$

(in the case $|S(r,s-1)\rangle^+ \to |S(r,s)\rangle^-$ we would have $\theta = -(s-1)t + r$ instead). Note that since there have been only the creation operators in the expression for $|S(r,s-1)\rangle^-$, we would not encounter $\mathcal{L}_0$ and $\mathcal{H}_0$, therefore $c$-number terms from (4.4) will not appear. The highest weight state has to be transformed accordingly:

$$|\mathsf{h}\rangle_{\rm top} \to g(\theta,-1)\,|\tfrac{2}{t} - \mathsf{h}\rangle_{\rm top}\,, \tag{7.3}$$

with the above $\theta$; in the case $|S(r,s-1)\rangle^+ \to |S(r,s)\rangle^-$ this would be $|\mathsf{h}\rangle_{\rm top} \to q(-\theta,-1)|\tfrac{2}{t} - 2 - \mathsf{h}\rangle_{\rm top}$. In this way, we obtain a singular vector in the generalized Verma module $\mathcal{V}^-_{r,s-1}$.

Step 2. To return to the module $V^+_{r,s}$ where we want to find a singular vector, apply to the expression resulting from Step 1 the intertwiner $g(-r,\theta-1)$ (respectively $q(-r,-\theta-1)$). In the resulting expression $g(-r,\theta-1)\,(\mathit{modes})\,g(\theta,-1)\,|\mathsf{h}^+(r,s)\rangle_{\rm top}$, we use relations (4.18) by commuting the modes of $\mathcal{G}$ on the left. This will eliminate *all* $\mathcal{G}$- and $\mathcal{Q}$- modes, leaving us with a polynomial in negative integral modes of $\mathcal{L}$ and $\mathcal{H}$ inside $g(-r,\theta-1)\,\dots\,g(\theta,-1)\,|\mathsf{h}^+(r,s)\rangle_{\rm top}$. This polynomial will be referred to as a 'skeleton' of the singular vector $|S(r,s)\rangle^+$.

Step 3. It remains to use relations (4.28) to commute the modes of $\mathcal{L}$ and $\mathcal{H}$ through the intertwiner $g(-r,\theta-1)$ so as to make the two intertwiners meet. To $g(-r+p+1,\theta-1)\,g(\theta,-1) = g(-r+p+1,-1)$ we apply (6.3) whenever possible, that is when $p+1 \le r$ (recall that both $p$ and $r$ are positive integers!). As a result, we will be left with two kinds of terms: i) those which contain no intertwiners,

$$\mathcal{P}(\mathcal{L},\mathcal{H})\,\mathcal{G}_{n_1}\,\dots\,\mathcal{G}_{n_r}\,|\mathsf{h}^+(r,s)\rangle_{\rm top}\,, \tag{7.4}$$

and are therefore in the Verma module $V^+_{r,s}$ ($\mathcal{P}$ is a polynomial in the negative modes and $n_i$ are negative integers), and ii) those involving intertwiners of *negative* integral length:

$$\mathcal{P}(\mathcal{L},\mathcal{H})\,\mathcal{G}_{n_1}\,\dots\,\mathcal{G}_{n_{r+N}}\,g(N,-1)\,|\mathsf{h}^+(r,s)\rangle_{\rm top}\,, \tag{7.5}$$

where $N = 1,2,\dots,r(s-2)$. Starting with higher $N$, we then replace

$$g(N,-1)\,|\mathsf{h}^+(r,s)\rangle_{\rm top} = \frac{t}{2(N-1)(\mathsf{h}^+(r,s)t - N)}\,\mathcal{Q}_{1-N}\,g(N-1,-1)\,|\mathsf{h}^+(r,s)\rangle_{\rm top}\,,\quad N \ge 2 \tag{7.6}$$

(which is a specification of (6.5), (6.6)) and commute the product $(\prod \mathcal{G})$ in (7.5) to the right [7]. Then, in some terms the combination $(\prod \mathcal{G})\, g(N-1,-1)$ would allow us to apply (4.17), which would give one of the $g(N-2,-1)$, $g(N-3,-1)$, ..., $g(0,-1)$ intertwiners. In the latter case ($g(0,-1)=1$) the intertwiner will have disappeared. In the terms that would still contain $g(N-1,-1)$, ..., $g(2,-1)$, we apply (7.6) and the corresponding rearrangements again, until we end up with having, on top of a state from the Verma module $V^+_{r,s}$, only the terms that contain $g(1,-1)$. However, *all the latter cancel against the different terms*, and we are therefore left with a state in $V^+_{r,s}$.

This state is the singular vector $|T(r,s)\rangle^+$, from which the 'conventional' singular vector follows as $|S(r,s)\rangle^+ = \mathcal{Q}_0 \ldots \mathcal{Q}_{r-1} \cdot |T(r,s)\rangle^+$.

The $|T(r,s)\rangle^-$ vector is derived similarly from $|S(r,s-1)\rangle^+$.

To make the construction of this section complete (sufficient for finding any topological singular vector), we have to quote the simplest $T$-vectors $|T(r,1)\rangle^{\pm}$:

$$\begin{aligned} |T(r,1)\rangle^+ &= \Big(\prod_{i=-r}^{-1} \mathcal{G}_i\Big)\, |\tfrac{1-r}{t}\rangle_{\rm top}\,, \\ |T(r,1)\rangle^- &= \Big(\prod_{i=-r}^{-1} \mathcal{Q}_i\Big)\, |-1+\tfrac{1+r}{t}\rangle_{\rm top}\,, \end{aligned} \qquad r=1,2\ldots \tag{7.7}$$

from which we construct $|S(r,1)\rangle^+$ as in (4.1) and similarly $|S(r,1)\rangle^-$.

In fact, the above recursive construction is *equivalent* to the formula (5.3) as soon as one notices that (5.3) is arrived at just by a successive application of the recursive prescriptions described in this section, starting from the $|T(r,1)\rangle$ vectors, which would then become precisely the product of modes of $\mathcal{G}$ or $\mathcal{Q}$ in the centre of the formula (5.3) as we have noted.

# 8 Concluding remarks

The formula (5.3) gives a general algebraic construction for the topological singular vectors. Comparison with the MFF construction shows that the topological case involves more complicated algebra. However, this is but a technical complication, since our construction, just like the MFF one, is 'direct' in that it does not require solving equations and it allows one to derive singular vectors from a generating expression by means of purely algebraic manipulations. The algebraic rules required in order to transform the generating expression to the conventional form (a Verma module element) consist of algebraic properties of the intertwining operators necessary for manipulations in the generalized Verma modules and mappings between different such modules.

The actual coincidence between the topological and affine $s\ell(2)$ singular vectors, observed in [18], should now be interpreted as the statement that the isomorphism between the two classes of singular vectors is supported by the two constructions – MFF and ours – *after* these constructions are 'continued' each in its own algebraic scheme. When $r$, $s$ and $t$ are chosen so that the formula for a topological singular vector does not require a 'continuation', the Kazama–Suzuki mapping can be applied to it

[7] Note that the normalization factor in (7.6) rewrites as

$$\frac{t}{2(N-1)(1-N-r+(s-1)t)}\,,$$

in which form it is equally applicable in a similar derivation of $|T(r,s)\rangle^-$ from $|S(r,s-1)\rangle^+$. Clearly, as was the case with formula (6.5), the use of (7.6) does never give rise to a rational dependence on $t$, since the terms to which (7.6) is applied are proportional to the respective factor $(1-N-r+(s-1)t)$.

directly and takes it precisely into the corresponding MFF singular vector, which in that case would be by itself (i.e. without any algebraic manipulations either) a Verma module element.

A lesson that may be drawn from the existence of two generating constructions is that general and at the same time closed formulae for singular vectors should probably be expected to have an 'algebraically continued' form. This is almost certainly so for the $N=2$ $W_3$ singular vectors, but it is still not clear how general this feature may actually be. For the Virasoro algebra, the question is particularly acute, since in the absence of any other operators except $L_n$, the algebra appears to be too 'poor' to allow 'algebraically extended' constructions that would allow for a monomial representation of singular vectors. However, an interesting fact is that the above derivation of topological singular vectors involved a stage (Step 2 in the previous section) when what would become a singular vector contained only $\mathcal{L}$ and $\mathcal{H}$ operators. That 'skeleton' form is therefore sufficient for the reconstruction of the entire topological singular vector[8]. On the other hand, one can reduce the topological singular vectors to those of the appropriate minimal matter theory [28, 22]. It would be interesting to see in detail how the general construction for Virasoro singular vectors is recovered along these lines, for instance by applying the reduction to the 'skeleton' form. More generally, there exist a class of theories (minimal matter, $s\ell(2)$, $N=2$, parafermions, ...) which are related by reductions (including the hamiltonian reduction) or embeddings (such as embeddings of minimal matter into $N=2$ [32, 33] or $s\ell(2)$ [18] theories), and which possess isomorphic (or 'almost' isomorphic, say in relation 2:1) singular vectors [21]. It would be interesting to see if different theories from that class provide new reformulations of the generating construction for essentially the same set of singular vectors.

It is also possible to extend the present construction to other $N=2$ singular vectors, which are built over non-chiral primary states [5]. The construction for all singular vectors of the $N=2$ superconformal algebra is surprisingly similar to the one described in this paper, and is given in [36].

Let us note finally that, in view of the appearance of twisted $N=2$ algebra in string theories [32, 28, 33], one may expect applications of the present construction to physical states [34] with non-trivial ghost numbers (the explicit form of MFF singular vectors has been used to construct cohomology e.g. in [35]).

We would like to thank B. Feigin for useful discussions. We are also grateful to B. Gato-Rivera, S. Hwang, W. Lerche, A. Pogrebkov, M.A. Soloviev, K. Stelle and Yu. Zinoviev for useful comments. This research was made possible in part by Grant No. MQM300 from the International Science Foundation and Government of Russian Federation, and by RFFI grant 94-02-06338-a.

## Appendix

Here we give two examples of the calculation following the scheme described in section 7. As a simple example, consider how the recursion procedure allows one to derive the singular vector $|T(1,3)\rangle^-$ out of

$$|S(1,2)\rangle^+ = \Big(-4t\mathcal{L}_{-2} + 4t\mathcal{H}_{-1}\mathcal{L}_{-1} + 4\mathcal{L}_{-1}\mathcal{L}_{-1} + 2t\mathcal{Q}_{-1}\mathcal{G}_{-1}\Big)|1\rangle_{\rm top} \tag{A.1}$$

We now apply the steps described in Section 7. After the spectral flow transform with parameter $1-2t$ we sandwich the expression in brackets in (A.1) between $q(-1,2t-2)\,\ldots\,q(2t-1,-1)|\frac{2}{t}-3\rangle_{\rm top}$ and then drop the term $2t\mathcal{Q}_{2t-2}\mathcal{G}_{-2t}$, in accordance with the '$q$'-version of (4.18). This leaves us with the

[8] Note that there is a certain 'dualism' in the roles played, on the one hand, by the fermionic generators $\mathcal{Q}$ and $\mathcal{G}$ and, on the other hand, by the bosonic ones $\mathcal{L}$ and $\mathcal{H}$. The generating formula (5.3) is, heuristically, expressed in terms of $\mathcal{Q}$ and $\mathcal{G}$ alone, while a crucial step in the course of its rearrangement produces a 'skeleton' of the desired singular vector, involving only $\mathcal{L}$ and $\mathcal{H}$; the 'new' $\mathcal{G}$ and $\mathcal{Q}$ modes are then reconstructed from the $\mathcal{L}$-$\mathcal{H}$-skeleton.

'skeleton' inside the intertwiners:

$$\begin{aligned}|T(1,3)\rangle^- &= q(-1,2t-2)\Big((-4+8t^2)\mathcal{H}_{-2}-4t\mathcal{L}_{-2}+(4-12t+8t^2)\mathcal{H}_{-1}\mathcal{H}_{-1}\\&\quad+(8-12t)\mathcal{L}_{-1}\mathcal{H}_{-1}+4\mathcal{L}_{-1}\mathcal{L}_{-1}\Big)\,q(2t-1,-1)\,|\tfrac{2}{t}-3\rangle_{\rm top}\end{aligned}\tag{A.2}$$

Here, we apply the '$q$'-analogue of the rules (4.28) (with $g\leftrightarrow q$ and $\mathcal{G}\leftrightarrow\mathcal{Q}$) to move the modes of $\mathcal{L}$ and $\mathcal{H}$ through the left intertwiner $q(-1,2t-2)$. Then the products of the intertwiners rewrite as

$$\begin{aligned}q(-1,2t-2)\,q(2t-1,-1)&=q(-1,-1)=\mathcal{Q}_{-1}\,,\\q(0,2t-2)\,q(2t-1,-1)&=q(0,-1)=1\,.\end{aligned}$$

The result is

$$\begin{aligned}|T(1,3)\rangle^- &= \Big((-4+8t^2)\mathcal{H}_{-2}\mathcal{Q}_{-1}+4t(-3+4t)\mathcal{H}_{-1}\mathcal{Q}_{-2}-4t\mathcal{L}_{-2}\mathcal{Q}_{-1}-12t\mathcal{L}_{-1}\mathcal{Q}_{-2}\\&\quad+(4-12t+8t^2)\mathcal{H}_{-1}\mathcal{H}_{-1}\mathcal{Q}_{-1}+(8-12t)\mathcal{L}_{-1}\mathcal{H}_{-1}\mathcal{Q}_{-1}+4\mathcal{L}_{-1}\mathcal{L}_{-1}\mathcal{Q}_{-1}\\&\quad+4t(1+4t)\mathcal{Q}_{-3}\Big)|\tfrac{2}{t}-3\rangle_{\rm top}\end{aligned}\tag{A.3}$$

which is the sought singular vector.

Consider now deriving the singular vector $|T(1,5)\rangle^+$ from $|S(1,4)\rangle^-$. The latter can be written as

$$\begin{aligned}|S(1,4)\rangle^- &= \Big((-6-36t-66t^2-36t^3)\mathcal{H}_{-4}+(-6t+24t^2-36t^3)\mathcal{L}_{-4}+(t-9t^2+18t^3)\mathcal{G}_{-3}\mathcal{Q}_{-1}\\&+(-3t^2+18t^3)\mathcal{G}_{-2}\mathcal{Q}_{-2}+(3t^2+18t^3)\mathcal{G}_{-1}\mathcal{Q}_{-3}+(8+16t-32t^2-48t^3)\mathcal{H}_{-3}\mathcal{H}_{-1}\\&+(3+6t-7t^2-18t^3)\mathcal{H}_{-2}\mathcal{H}_{-2}+(-10t+39t^2-36t^3)\mathcal{L}_{-3}\mathcal{H}_{-1}+(-10t+24t^2)\mathcal{L}_{-3}\mathcal{L}_{-1}\\&+(10t+12t^2-18t^3)\mathcal{L}_{-2}\mathcal{H}_{-2}+9t^2\mathcal{L}_{-2}\mathcal{L}_{-2}+(8+28t+13t^2-12t^3)\mathcal{L}_{-1}\mathcal{H}_{-3}\\&+(2t-14t^2+18t^3)\mathcal{G}_{-2}\mathcal{H}_{-1}\mathcal{Q}_{-1}+(2t-\tfrac{21}{2}t^2)\mathcal{G}_{-2}\mathcal{L}_{-1}\mathcal{Q}_{-1}+(-3t-\tfrac{3}{2}t^2+9t^3)\mathcal{G}_{-1}\mathcal{H}_{-2}\mathcal{Q}_{-1}\\&+(-\tfrac{21}{2}t^2+18t^3)\mathcal{G}_{-1}\mathcal{H}_{-1}\mathcal{Q}_{-2}-\tfrac{9}{2}t^2\mathcal{G}_{-1}\mathcal{L}_{-2}\mathcal{Q}_{-1}-\tfrac{21}{2}t^2\mathcal{G}_{-1}\mathcal{L}_{-1}\mathcal{Q}_{-2}\\&-(6-12t-22t^2+36t^3)\mathcal{H}_{-2}\mathcal{H}_{-1}\mathcal{H}_{-1}-(10t-30t^2+18t^3)\mathcal{L}_{-2}\mathcal{H}_{-1}\mathcal{H}_{-1}+3t\mathcal{G}_{-1}\mathcal{L}_{-1}\mathcal{L}_{-1}\mathcal{Q}_{-1}\\&+(-20t+30t^2)\mathcal{L}_{-2}\mathcal{L}_{-1}\mathcal{H}_{-1}-10t\mathcal{L}_{-2}\mathcal{L}_{-1}\mathcal{L}_{-1}+(-12+6t+43t^2-18t^3)\mathcal{L}_{-1}\mathcal{H}_{-2}\mathcal{H}_{-1}\\&-(6+6t-10t^2)\mathcal{L}_{-1}\mathcal{L}_{-1}\mathcal{H}_{-2}+(3t-11t^2+9t^3)\mathcal{G}_{-1}\mathcal{H}_{-1}\mathcal{H}_{-1}\mathcal{Q}_{-1}+(6t-11t^2)\mathcal{G}_{-1}\mathcal{L}_{-1}\mathcal{H}_{-1}\mathcal{Q}_{-1}\\&+(1-6t+11t^2-6t^3)\mathcal{H}_{-1}\mathcal{H}_{-1}\mathcal{H}_{-1}\mathcal{H}_{-1}+(4-18t+22t^2-6t^3)\mathcal{L}_{-1}\mathcal{H}_{-1}\mathcal{H}_{-1}\mathcal{H}_{-1}\\&+(6-18t+11t^2)\mathcal{L}_{-1}\mathcal{L}_{-1}\mathcal{H}_{-1}\mathcal{H}_{-1}+(4-6t)\mathcal{L}_{-1}\mathcal{L}_{-1}\mathcal{L}_{-1}\mathcal{H}_{-1}+\mathcal{L}_{-1}\mathcal{L}_{-1}\mathcal{L}_{-1}\mathcal{L}_{-1}\Big)|-4+\tfrac{2}{t}\rangle_{\rm top}\end{aligned}\tag{A.4}$$

where we have chosen the ordering with $\mathcal{G}$ modes on the left. Then, after the spectral flow transform $\mathcal{G}_n\mapsto\mathcal{G}_{n+4t-1}$, the modes of $\mathcal{G}$ will be immediately killed in accordance with (4.18) when multiplied from the left with the intertwiner $g(-1,4t-2)$. Performing in the remaining ($\mathcal{Q}$- and $\mathcal{G}$- independent) terms the substitution $\mathcal{L}_n\mapsto\mathcal{L}_n+(4t-1)\mathcal{H}_n$, will then give the $\mathcal{L}$-$\mathcal{H}$-skeleton, and the $|T(1,5)\rangle^+$ vector becomes $g(-1,4t-2)\cdot(\mathit{skeleton})\cdot g(4t-1,-1)|4\rangle_{\rm top}$. Next, commuting the modes of $\mathcal{L}$ and $\mathcal{H}$ from the skeleton through the intertwiner $g(-1,4t-2)$ according to the formulae (4.28), and using (4.16) and (6.3), we find the following terms which contain no $g(N,-1)$ factors of negative length,

$$\begin{aligned}&\Big((6t+83t^2+380t^3+576t^4)\mathcal{G}_{-5}-(24t+144t^2+264t^3+144t^4)\mathcal{H}_{-4}\mathcal{G}_{-1}\\&-(72t^2+256t^3+192t^4)\mathcal{H}_{-3}\mathcal{G}_{-2}-(30t^2+242t^3+288t^4)\mathcal{H}_{-2}\mathcal{G}_{-3}+(25t^2-60t^3)\mathcal{L}_{-3}\mathcal{G}_{-2}\\&-(22t^2+228t^3+576t^4)\mathcal{H}_{-1}\mathcal{G}_{-4}-(6t-24t^2+36t^3)\mathcal{L}_{-4}\mathcal{G}_{-1}+(32t^2+96t^3+72t^4)\mathcal{H}_{-2}\mathcal{H}_{-2}\mathcal{G}_{-1}\\&-(10t+106t^2+272t^3)\mathcal{L}_{-1}\mathcal{G}_{-4}+(72t^2+256t^3+192t^4)\mathcal{H}_{-3}\mathcal{H}_{-1}\mathcal{G}_{-1}-(16t^2+112t^3)\mathcal{L}_{-2}\mathcal{G}_{-3}\end{aligned}$$

$$
\begin{aligned}
&+ (208t^3 + 288t^4)\mathcal{H}_{-2}\mathcal{H}_{-1}\mathcal{G}_{-2} + (46t^3 + 288t^4)\mathcal{H}_{-1}\mathcal{H}_{-1}\mathcal{G}_{-3} + (-25t^2 + 60t^3)\mathcal{L}_{-3}\mathcal{H}_{-1}\mathcal{G}_{-1}\\
&+ (-10t + 24t^2)\mathcal{L}_{-3}\mathcal{L}_{-1}\mathcal{G}_{-1} + (34t^2 + 54t^3)\mathcal{L}_{-2}\mathcal{H}_{-2}\mathcal{G}_{-1} + 116t^3\mathcal{L}_{-2}\mathcal{H}_{-1}\mathcal{G}_{-2} + 9t^2\mathcal{L}_{-2}\mathcal{L}_{-2}\mathcal{G}_{-1}\\
&+ 50t^2\mathcal{L}_{-2}\mathcal{L}_{-1}\mathcal{G}_{-2} + (30t + 109t^2 + 84t^3)\mathcal{L}_{-1}\mathcal{H}_{-3}\mathcal{G}_{-1} + (95t^2 + 138t^3)\mathcal{L}_{-1}\mathcal{H}_{-2}\mathcal{G}_{-2} \qquad \text{(A.5)}\\
&+ (45t^2 + 288t^3)\mathcal{L}_{-1}\mathcal{H}_{-1}\mathcal{G}_{-3} + (10t + 65t^2)\mathcal{L}_{-1}\mathcal{L}_{-1}\mathcal{G}_{-3} - (104t^3 + 144t^4)\mathcal{H}_{-2}\mathcal{H}_{-1}\mathcal{H}_{-1}\mathcal{G}_{-1}\\
&- 96t^4\mathcal{H}_{-1}\mathcal{H}_{-1}\mathcal{H}_{-1}\mathcal{G}_{-2} - 58t^3\mathcal{L}_{-2}\mathcal{H}_{-1}\mathcal{H}_{-1}\mathcal{G}_{-1} - 50t^2\mathcal{L}_{-2}\mathcal{L}_{-1}\mathcal{H}_{-1}\mathcal{G}_{-1} - 10t\mathcal{L}_{-2}\mathcal{L}_{-1}\mathcal{L}_{-1}\mathcal{G}_{-1}\\
&- (95t^2 + 138t^3)\mathcal{L}_{-1}\mathcal{H}_{-2}\mathcal{H}_{-1}\mathcal{G}_{-1} - 150t^3\mathcal{L}_{-1}\mathcal{H}_{-1}\mathcal{H}_{-1}\mathcal{G}_{-2} - (20t + 30t^2)\mathcal{L}_{-1}\mathcal{L}_{-1}\mathcal{H}_{-2}\mathcal{G}_{-1}\\
&+ 24t^4\mathcal{H}_{-1}\mathcal{H}_{-1}\mathcal{H}_{-1}\mathcal{H}_{-1}\mathcal{G}_{-1} + 50t^3\mathcal{L}_{-1}\mathcal{H}_{-1}\mathcal{H}_{-1}\mathcal{H}_{-1}\mathcal{G}_{-1} + 35t^2\mathcal{L}_{-1}\mathcal{L}_{-1}\mathcal{H}_{-1}\mathcal{H}_{-1}\mathcal{G}_{-1}\\
&- 70t^2\mathcal{L}_{-1}\mathcal{L}_{-1}\mathcal{H}_{-1}\mathcal{G}_{-2} - 10t\mathcal{L}_{-1}\mathcal{L}_{-1}\mathcal{L}_{-1}\mathcal{G}_{-2} + 10t\mathcal{L}_{-1}\mathcal{L}_{-1}\mathcal{L}_{-1}\mathcal{H}_{-1}\mathcal{G}_{-1}\\
&+ \mathcal{L}_{-1}\mathcal{L}_{-1}\mathcal{L}_{-1}\mathcal{L}_{-1}\mathcal{G}_{-1}\Big)\,|4\rangle_{\rm top}
\end{aligned}
$$

as well as 13 terms that do contain negative-length intertwiners:

$$
\begin{aligned}
&\Big((13t^2 + 56t^3)\mathcal{G}_{-4}\mathcal{G}_{-1}\,g(1,-1) + (t^2 + 12t^3)\mathcal{G}_{-3}\mathcal{G}_{-2}\,g(1,-1) - (6t + 26t^2 - 52t^3)\mathcal{G}_{-3}\mathcal{G}_{-1}\mathcal{G}_0\,g(2,-1)\\
&- (4t^2 + 20t^3)\mathcal{H}_{-2}\mathcal{G}_{-2}\mathcal{G}_{-1}\,g(1,-1) - (3t^2 + 40t^3)\mathcal{H}_{-1}\mathcal{G}_{-3}\mathcal{G}_{-1}\,g(1,-1)\\
&- (13t^2 + 16t^3)\mathcal{L}_{-1}\mathcal{G}_{-3}\mathcal{G}_{-1}\,g(1,-1) + (12t + 8t^2 - 32t^3)\mathcal{G}_{-2}\mathcal{G}_{-1}\mathcal{G}_0\mathcal{G}_1\,g(3,-1)\\
&+ (28t^2 - 40t^3)\mathcal{H}_{-1}\mathcal{G}_{-2}\mathcal{G}_{-1}\mathcal{G}_0\,g(2,-1) + 12t^3\mathcal{H}_{-1}\mathcal{H}_{-1}\mathcal{G}_{-2}\mathcal{G}_{-1}\,g(1,-1)\\
&+ (10t - 18t^2 + 4t^3)\mathcal{L}_{-1}\mathcal{G}_{-2}\mathcal{G}_{-1}\mathcal{G}_0\,g(2,-1) + (5t^2 + 12t^3)\mathcal{L}_{-1}\mathcal{H}_{-1}\mathcal{G}_{-2}\mathcal{G}_{-1}\,g(1,-1)\\
&- (2t^2 + 4t^3)\mathcal{L}_{-2}\mathcal{G}_{-2}\mathcal{G}_{-1}\,g(1,-1) + 5t^2\mathcal{L}_{-1}\mathcal{L}_{-1}\mathcal{G}_{-2}\mathcal{G}_{-1}\,g(1,-1)\Big)\,|4\rangle_{\rm top} \qquad \text{(A.6)}
\end{aligned}
$$

To 'resolve' the negative-length intertwiners, we proceed as in Sect. 7. First, $\mathcal{G}_{-2}\mathcal{G}_{-1}\mathcal{G}_0\mathcal{G}_1\,g(3,-1)\,|4\rangle_{\rm top}$ rewrites as $\frac{t}{4(4t-3)}\,\mathcal{G}_{-2}\mathcal{G}_{-1}\mathcal{G}_0\mathcal{G}_1\mathcal{Q}_{-2}\,g(2,-1)\,|4\rangle_{\rm top}$. Then, commuting $\mathcal{G}_1$ through $\mathcal{Q}_{-2}$, we would get $\mathcal{G}_1\,g(2,-1)\,|4\rangle_{\rm top} = g(1,-1)\,|4\rangle_{\rm top}$. To all the remaining terms that contain $g(2,-1)\,|4\rangle_{\rm top}$ we apply the formula (7.6) again, and thus (A.6) becomes

$$
\begin{aligned}
&\Big((8t^2 + 36t^3)\mathcal{G}_{-5} + (7t^2 + 36t^3)\mathcal{H}_{-4}\mathcal{G}_{-1} - (4t^2 + 20t^3)\mathcal{H}_{-3}\mathcal{G}_{-2} - (3t^2 + 16t^3)\mathcal{H}_{-2}\mathcal{G}_{-3}\\
&- 12t^3\mathcal{H}_{-1}\mathcal{G}_{-4} + (-5t^2 + 20t^3)\mathcal{L}_{-4}\mathcal{G}_{-1} + (3t^2 - 4t^3)\mathcal{L}_{-3}\mathcal{G}_{-2} - (3t^2 + 16t^3)\mathcal{L}_{-2}\mathcal{G}_{-3}\\
&- 5t^2\mathcal{L}_{-1}\mathcal{G}_{-4} - 12t^3\mathcal{H}_{-3}\mathcal{H}_{-1}\mathcal{G}_{-1} + 12t^3\mathcal{H}_{-2}\mathcal{H}_{-1}\mathcal{G}_{-2} - 12t^3\mathcal{L}_{-3}\mathcal{H}_{-1}\mathcal{G}_{-1} - 5t^2\mathcal{L}_{-3}\mathcal{L}_{-1}\mathcal{G}_{-1}\\
&+ 12t^3\mathcal{L}_{-2}\mathcal{H}_{-1}\mathcal{G}_{-2} + 5t^2\mathcal{L}_{-2}\mathcal{L}_{-1}\mathcal{G}_{-2} - 5t^2\mathcal{L}_{-1}\mathcal{H}_{-3}\mathcal{G}_{-1} + 5t^2\mathcal{L}_{-1}\mathcal{H}_{-2}\mathcal{G}_{-2} - (t^2 + 2t^3)\mathcal{Q}_{-2}\mathcal{G}_{-2}\mathcal{G}_{-1}\\
&- (\tfrac{3}{2}t^2 + 8t^3)\mathcal{Q}_{-1}\mathcal{G}_{-3}\mathcal{G}_{-1} + 6t^3\mathcal{H}_{-1}\mathcal{Q}_{-1}\mathcal{G}_{-2}\mathcal{G}_{-1} + \tfrac{5}{2}t^2\mathcal{L}_{-1}\mathcal{Q}_{-1}\mathcal{G}_{-2}\mathcal{G}_{-1}\Big)\,|4\rangle_{\rm top} \qquad \text{(A.7)}
\end{aligned}
$$

Adding these terms with (A.5) does give the singular vector $|T(1,5)\rangle^+$.

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